WAVE IMPACT ON THE CENTER OF AN EULER BEAM

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The problem of a symmetric wave impact on the Euler beam is solved by the normal modes method. The liquid is supposed to be ideal and incompressible. The initial stage of impact when hydrodynamic loads are very high and the beam is wetted only partially is considered. The flow of a liquid and the size of the wetted part of the body are determined by the Wagner approach with a simultaneous calculation of the beam deflection. The specific features of the developed numerical algorithm are demonstrated and the criterion of its stability is specified. In addition to a direct solution of the problem, two approximate approaches within the framework of which the dimension of the contact region is found ignoring the deformations of the plate are considered.

Introduction. The plane unsteady problem of an impact by an ideal incompressible liquid on an elastic plate of finite length is considered. At the initial moment ($t' = 0$), a weakly curved boundary of the liquid touches the plate in its center ($x' = 0$ and $y' = 0$), and the velocity of all liquid particles is equal to V and is directed along the normal toward the undeformed surface of the plate (Fig. 1a). The initial position of the free boundary is assumed to be symmetric about the Oy' axis. The lower side of the plate is plane, and the transverse cross section of the plate does not depend on the longitudinal coordinate x'. The maximum thickness of the plate h is assumed to be much less than its length $2L$ and the width of the plane side b. The plate is simply supported at its ends, and its deflection is governed by the Euler beam equation. Only the bending stresses in the longitudinal direction are taken into consideration in the Euler model. The shear stresses and the stresses in the transverse direction are assumed to be small. For $t' > 0$, the liquid strikes the plate. The impact stage, during which hydrodynamic loads are very significant, finishes at the moment when the plate is completely wetted. For a weakly curved liquid boundary, which corresponds to the impact on a catamaran wetdeck by a sufficiently long wave, the impact stage is short.

The hydrodynamic loads on impact increase rapidly with time and then damp. To estimate the duration of the impact stage, we note that the initial shape of the free boundary near its top can be approximated by the parabolic contour $y' = -x'^2/(2R)$, where R is the radius of curvature of the undisturbed liquid boundary at the origin of the Cartesian coordinate system. The dimensional variables are primed. For a weakly curved free boundary, the ratio $\varepsilon = L/R$ is small. Ignoring the deformation of the liquid boundary on impact and assuming the vertical wave velocity to be constant, we expect that the plate will be completely wetted at the moment T_1, when $-L^2/(2R) + VT_1 = 0$. In the impact stage, the quantity $T = L^2/(RV)$ is used as the scale of time. The actual duration of the stage is less owing to an additional rise of the free boundary toward the plate (Fig. 1b) but it is of the same order as T.

The distinguishing feature of the problem is that the elastic plate is deformed by hydrodynamic loads, the region of application of which $-c'(t') < x' < c'(t')$ extends with time, and their amplitude itself depends on the beam deflection. The problem is coupled: in the general case, the liquid flow and the deformations of a body should be determined simultaneously. At the same time, it is necessary to determine the dimension of the wetted part of the body, which is the important characteristic of the process. The calculation of the

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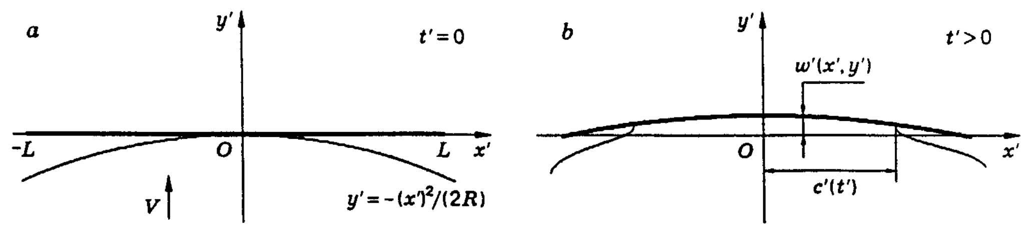

Fig. 1

function $c'(t')$ entails significant difficulties and usually is approximate in character [1]. In the present work, the method of simultaneously calculating the hydrodynamic (the velocity field and the pressure distribution), elastic (bending stresses and deflections), and geometric (the dimension of the contact region and the shape of the free boundary) characteristics of the impact of elastic bodies on a liquid is proposed.

It is required to determine the deformation of the elastic plate, the distribution of bending stresses, the pressure distribution along the contact region, and the position of the contact points under the following assumptions:

1) The liquid is ideal and incompressible;

2) The liquid flow is plane, potential, and symmetric about the Oy' axis;

3) The initial radius of curvature of the free boundary at the contact point R is much larger, and the maximum thickness of the plate h is much less than its length $2L$;

4) The plate is governed by the Euler beam equation, and its end points are simply supported;

5) The external mass forces and the forces of surface tension are absent.

The scales of variation of the hydrodynamic variables are chosen the same as for an undeformable plate: L is the scale of length, V is the scale of velocity, VL is the scale of velocity potential, $\rho VL/T$ is the scale of pressure, where ρ is the density of the liquid and the scale of time T has been determined above. The scale of beam deflection W is not fixed beforehand. The instructions on its choice will be given below. The scale of bending stresses in the plate is taken to be equal to hEW/L^2, where E is Young's modulus. In what follows, the dimensionless variables, which are not primed, are used.

Formulation of the Problem. The problem is conveniently analyzed with the use of a moving deformed coordinate system x_1, y_1 such that $x_1 = x$ and $y_1 = y + \varepsilon(x^2/2 - t)$ and to present the velocity potential $\varphi(x,y,t)$ in the form $\varphi(x,y,t) = y - t^2/2 + \varphi_1(x_1,y_1,t)$, where $\varphi_1(x_1,y_1,t)$ is the potential of the perturbed motion of the liquid on impact. In the new coordinate system, the initial position of the free boundary corresponds to the horizontal line $y_1 = 0$, and the position of the elastic plate is described by the equation $y_1 = \varepsilon y_B(x_1,t)$, where $y_B(x_1,t) = x_1^2/2 + æ w(x_1,t) - t$ and $æ = W/(L\varepsilon)$, and the function $w(x,t)$ sets the amplitude of the elastic deflection of the plate in a point with the x coordinate at the moment t. The equation of motion of the fluid and the boundary conditions acquire a more complicated form; however, it is easy to verify that all the new and nonlinear terms in these equations have the coefficient ε. This allows us to ignore them formally under the condition that $\varepsilon \ll 1$. The discarded terms have the order $O(\varepsilon)$ almost everywhere, except the periphery of the contact region, where it is necessary to construct an "internal" asymptotic expansion of the solution with allowance for nonlinear effects. With the same accuracy, the boundary conditions at the liquid boundary can be linearized and imposed on its unperturbed level $y_1 = 0$. In the first approximation, as $\varepsilon \to 0$ the motion of the liquid is described almost everywhere by the Laplace equation for the velocity potential $\varphi_1(x_1,y_1,t)$ in the lower half-plane $y_1 < 0$, and the deformations of the plate by the Euler equation relative to the beam deflection $w(x_1,t)$ [2]. In the symmetric case, the position of the contact points is set by a single function $c(t)$. Despite the fact that the equations of motion and the boundary conditions are linearized, the problem remains nonlinear, because the quantity $c(t)$ is unknown beforehand. Precisely the last circumstance determines difficulties which arise in a study of the impact of elastic bodies on the liquid. Below, the subscript 1 is omitted. Since the map $(x,y) \leftrightarrow (x_1,y_1)$ is identical with accuracy $O(\varepsilon)$, within which the problem is solved, the absence of the subscript should not bring about the misunderstanding. As $\varepsilon \to 0$, the identity of the map in the principal order means that in the collision of weakly curved surfaces, most important is the magnitude of the gap between them at the initial moment, rather than particular forms of each surface [3].

The formulation of the problem in dimensionless variables has the form

$$\alpha \frac{\partial^2 w}{\partial t^2} + \beta \frac{\partial^4 w}{\partial x^4} = p(x,0,t) \qquad (|x| < 1,\ t > 0); \tag{1}$$

$$w = w_{xx} = 0 \qquad (x = \pm 1,\ t \geqslant 0); \tag{2}$$

$$w = w_t = 0 \qquad (|x| \leqslant 1,\ t = 0); \tag{3}$$

$$p = -\varphi_t \qquad (y \leqslant 0); \tag{4}$$

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad (y < 0); \tag{5}$$

$$\varphi = 0 \qquad [y = 0,\ |x| > c(t)]; \tag{6}$$

$$\varphi_y = -1 + æ w_t(x,t) \qquad [y = 0,\ |x| < c(t)]; \tag{7}$$

$$\varphi \to 0 \qquad (x^2 + y^2 \to \infty). \tag{8}$$

Here $p(x,y,t)$ is the pressure of the liquid; the distribution of the bending stresses in the beam $\sigma(x,t)$ is taken to be linear in its thickness and is determined in dimensionless variables by the equality $\sigma(x,t) = z w_{xx}(x,t)/2$, where the variable z changes over the beam thickness, $z = -1$ corresponds to the lower wetted side, and $z = +1$ to its upper side in the largest-thickness sites. The upper side of the beam is compressed for $w_{xx}(x,t) > 0$ and stretches for $w_{xx}(x,t) < 0$. The formula for the pressure (4) follows from the linearized Cauchy–Lagrange integral. The dimensionless parameters α and β in the beam equation (1) and $æ$ in the condition in the contact region (7) are as follows:

$$\alpha = \frac{M_B}{\rho L} æ, \qquad \beta = \frac{EJ}{\rho L (RV)^2} æ, \qquad æ = \frac{RW}{L^2}, \tag{9}$$

where M_B and J is, respectively, the bram mass per unit length and the moment of inertia of the beam cross section, which are referred to the width of the lower plane side of the beam b. It is convenient to choose the scale of beam deflection W, so that one of the parameters in (9) is equal to and the other two do not exceed unity. For example, in the case considered in [4] for the aluminum wetdeck of a catamaran, it is assumed that $L = 75$ cm, $E = 7 \cdot 10^7$ N/m^2, $J = 1.106 \cdot 10^{-5}$ m^3, $M_B = 36.6$ kg/m^2, $V = 6$ m/sec, $\rho = 1000$ kg/m^3, $R = 40$ m, and $h = 12$ cm. We have $æ/\alpha = 20.5$ and $\beta/\alpha = 0.3672$, and hence $\beta < \alpha < æ$ with any choice of the scale W. We choose W such that $æ = 1$, and hence $W = L^2/R$. Here $\alpha = 4.88 \cdot 10^{-2}$ and $\beta = 1.8 \cdot 10^{-2}$. It is seen that the parameters α and β are small; they have the same order of smallness as the linearization parameter ε, which is equal to 0.01875 in this case. This points to the possibility of further simplification of system (1)–(8) by the methods of asymptotic analysis. Under different impact conditions, the three parameters can be of the order of unity, and problem (1)–(8) will be, therefore, considered below under the assumption that $\alpha = O(1)$, $\beta = O(1)$, and $æ = O(1)$.

Problem (1)–(8) should be supplemented by two conditions, which have the character of one-sided inequalities. The first condition is employed to determine the function $c(t)$ and implies that the liquid particles cannot penetrate an elastic plate. In the symmetrical case, this condition leads to the simple transcendental equation [5]

$$\int_0^{\pi/2} y_b[c(t)\sin\theta, t]\, d\theta = 0, \tag{10}$$

where the function $y_b(x,t)$ describes the shape of the elastic surface in a moving deformed coordinate system.

In our case, $y_b(x,t) = x^2/2 - t + æw(x,t)$, and Eq. (10) gives

$$t = c^2/4 + (2æ/\pi) \int_0^{\pi/2} w[c(t)\sin\theta, t]\, d\theta. \tag{11}$$

It is possible to show that Eq. (10) is equivalent to the well-known Wagner condition [6] in the problem of an impact from a blunt body on the free surface of a liquid. However, the last gives rise to a singular integral equation relative to the function $c(t)$. The solution of this equation entails serious computational difficulties even for the case of an undeformable body. Therefore, the dimension of the wetted part of the rigid surface is usually determined approximately within the framework of the linearized model (1)–(8) [1].

The second condition consists in that the pressure in the contact region cannot drop to the limiting value p_*. In the points of the contact region, where the pressure drops to the limiting value, the liquid detaches from the elastic surface with the formation of cavities filled with the liquid vapor. Cavitational phenomena in the water impact on elastic bodies were really observed in experiments [7] and can change significantly the distribution of hydrodynamic loads and the character of deformations. However, the solution of impact problems in the presence of unknown «internal» free boundaries faces great difficulties. In view of this, another approach is proposed, within the framework of which the condition of the admissible lower value of the pressure is omitted, but the value of the pressure in the contact region is controlled during computations. It is assumed that $p_* = 0$. If the pressure is negative on the major part of the contact region, the subsequent computation results are considered inadequate. This approach is based on the assumption that immediately after the moment of impact, the pressure in the contact region is positive and can become negative later only owing to the plate flexibility. The positions of the points of negative pressure and the velocities of the expansion of the negative-pressure regions is of doubtless interest and can be useful for the construction of complicated models that incorporate cavitational phenomena.

Normal Modes Method. A numerical study of problem (1)–(8) and (11) is based on the normal modes method. Within the framework of this method, the beam deflection is sought in the form

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x), \tag{12}$$

where $\psi_n(x)$ are the nontrivial solutions of the homogeneous boundary-value problem

$$\frac{d^4\psi_n}{dx^4} = \lambda_n^4 \psi_n \qquad (-1 < x < 1), \tag{13}$$

$$\psi_n = \frac{d^2\psi_n}{dx^2} = 0 \qquad (x = \pm 1) \tag{14}$$

(λ_n are the corresponding eigenvalues). The eigenfunctions $\psi_n(x)$ satisfy the orthogonality condition

$$\int_{-1}^{1} \psi_n(x)\psi_m(x)\, dx = \delta_{nm}, \tag{15}$$

where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$. Generally, the form of the functions $\psi_n(x)$ is cumbersome; however, under a symmetrical external load and with the simply supported ends of the beam Eqs. (13)–(15) produce the simple relations $\psi_n(x) = \cos \lambda_n x$ and $\lambda_n = \pi(n - 1/2)$. It is convenient to take the generalized coordinates of the modes $a_n(t)$, where $n = 1, 2, \ldots$, as the new unknown functions and to express other quantities via them.

On the section of the liquid boundary $-1 < x < 1$, $y = 0$, which corresponds to the contact region, the velocity potential and the pressure distribution can be presented as follows:

$$\varphi(x,0,t) = \sum_{n=1}^{\infty} b_n(t)\psi_n(x), \qquad p(x,0,t) = -\sum_{n=1}^{\infty} \dot{b}_n(t)\psi_n(x); \tag{16}$$

$$b_n(t) = \int_{-c(t)}^{c(t)} \varphi(x,0,t)\psi_n(x)\,dx, \tag{17}$$

as a consequence of (4), (6), and (15). The dot denotes the derivative in time. To find the dependences $b_m(t)$ on the generalized coordinate $a_n(t)$, where $m, n = 1, 2, \ldots$, we consider the hydrodynamic part of problem (1)–(8) separately.

We define the new, harmonic in the lower half-plane, functions $\varphi_n(x, y, c)$ as the solutions of the boundary-value problem

$$\frac{\partial^2\varphi_n}{\partial x^2} + \frac{\partial^2\varphi_n}{\partial y^2} = 0 \qquad (y < 0); \tag{18}$$

$$\varphi_n = 0 \qquad [y = 0,\ |x| > c(t)]; \tag{19}$$

$$\frac{\partial\varphi_n}{\partial y} = \psi_n(x) \qquad [y = 0,\ |x| < c(t)]; \tag{20}$$

$$\varphi_n \to 0 \qquad (x^2 + y^2 \to \infty) \tag{21}$$

with integrable singularities of the first derivatives near the points of change of the form of the boundary condition, $x = \pm c$. Here $n = 0, 1, 2, \ldots$ and $\psi_0(x) \equiv 1$. Taking into account that if the function $c(t)$ is known, problems (1)–(8) and (18)–(21) are linear, and comparing the boundary conditions (7) and (20), we obtain

$$\varphi(x,0,t) = -\varphi_0(x,0,c) + æ\sum_{n=1}^{\infty}\dot a_n(t)\varphi_n(x,0,c), \quad b_m(t) = -f_m(c) + æ\sum_{n=1}^{\infty}\dot a_n(t)S_{nm}(c). \tag{22}$$

Here

$$f_m(c) = \int_{-c}^{c}\varphi_0(x,0,c)\psi_m(x)\,dx, \qquad S_{nm}(c) = \int_{-c}^{c}\varphi_n(x,0,c)\psi_m(x)\,dx. \tag{23}$$

It is noteworthy that the matrix S with the elements $S_{nm}(c)$, where $m, n = 1, 2, \ldots$, is symmetric, which follows from (20) and (23) and the second integral of the Green theorem and depends only on the dimension of the contact region c. It is known [8] that $\varphi_0(x, 0, c) = \sqrt{c^2 - x^2}$ and $|x| < c$, whence $f_m(c) = \pi c^2 J_1(\lambda_m c)/(\lambda_m c)$ in the case of a simply supported beam.

Substituting the representations of the beam deflection (12) and of the hydrodynamic pressure (16) in the beam equation (1) and taking into consideration the orthogonality condition (15), Eq. (13), and formula (22), we obtain the infinite system of ordinary differential equations relative to the generalized coordinate:

$$\frac{d\mathbf{a}}{dt} = (\alpha I + æS)^{-1}(\beta D\mathbf{d} + \mathbf{f}); \tag{24}$$

$$\frac{d\mathbf{d}}{dt} = -\mathbf{a}. \tag{25}$$

Here $\mathbf{a} = (a_1, a_2, a_3, \ldots)^t$; $\mathbf{d}$ the additional vector, $\mathbf{d} = (d_1, d_2, d_3, \ldots)^t$, $d_n = (\beta\lambda_n^4)^{-1}(\alpha\dot a_n + b_n)$; $\mathbf{f} = (f_1(c), f_2(c), f_3(c), \ldots)^t$; I is a unit matrix, and D is a diagonal matrix, $D = \mathrm{diag}\{\lambda_1^4, \lambda_2^4, \lambda_3^4, \ldots\}$. The right-hand side of system (24) and (25) depends on $\mathbf{a}$, $\mathbf{d}$, and c and does not depend on t. Therefore, it is convenient to take the quantity c as a new independent variable with $0 \leqslant c \leqslant 1$. The differential equation for $t = t(c)$ follows from (11), if one differentiates this equation with respect to c:

$$\frac{dt}{dc} = Q(c, \mathbf{a}, \dot{\mathbf{a}}), \tag{26}$$

where

$$Q(c, \mathbf{a}, \dot{\mathbf{a}}) = \frac{c + (4æ/\pi)(\mathbf{a}, \mathbf{\Gamma}_c(c))}{2 - (4æ/\pi)(\dot{\mathbf{a}}, \mathbf{\Gamma}(c))}. \tag{27}$$

Here $(\mathbf{a}, \mathbf{b})$ is the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{\Gamma}(c) = (\Gamma_1(c), \Gamma_2(c), \ldots)$, $\mathbf{\Gamma}_c(c) = (\Gamma_{1c}(c), \Gamma_{2c}(c), \Gamma_{3c}(c), \ldots)$,

$$\Gamma_n(c) = \int\limits_0^{\pi/2} \psi_n(c\sin\theta)\, d\theta, \qquad \Gamma_{nc}(c) = \int\limits_0^{\pi/2} \psi_n'(c\sin\theta)\sin\theta\, d\theta.$$

Multiplying each equation of system (24) and (25) by dt/dc and taking into account (26), we find

$$\frac{d\mathbf{a}}{dc} = \mathbf{F}(c, \mathbf{d}) Q(c, \mathbf{a}, \mathbf{F}(c, \mathbf{d})); \tag{28}$$

$$\frac{d\mathbf{d}}{dc} = -\mathbf{a} Q(c, \mathbf{a}, \mathbf{F}(c, \mathbf{d})). \tag{29}$$

Here $\mathbf{F}(c, \mathbf{d}) = (\alpha I + æS(c))^{-1}(\beta D\mathbf{d} + \mathbf{f}(c))$. System (26)–(29) is solved numerically under the zero initial conditions

$$\mathbf{a} = 0, \qquad \mathbf{d} = 0, \qquad t = 0 \quad (c = 0). \tag{30}$$

It seems quite natural to choose the quantity c as an independent variable, because it corresponds to the structure of system (24) and (25). Introduction of the new unknown functions $d_n(t)$ instead of the derivatives $\dot{a}_n(t)$, where $n = 1, 2, 3, \ldots$, solves the problem of the beginning of a numerical calculation: the right-hand parts in system (26), (28), and (29) are equal to zero for $c = 0$. If the problem is solved in the initial variables, there are difficulties with the beginning of the calculation which are overcome if artificial methods are used [4, 9]. The reason is that, for short times, $c(t) = O(\sqrt{t})$, $w(x,t) = O(t^{3/2})$, $w_t = O(\sqrt{t})$, and $w_{tt} = O(t^{-1/2})$, i.e., at the beginning of the impact, the contact region extends with a very large velocity, and the accelerations of the elastic elements of the beam are unbounded. On the other hand, $t = O(c^2)$, $w = O(c^3)$, $w_t = O(c)$, and $w_{tt} = O(c^{-1})$, but $d_n = O(c^5)$ as $c \to 0$. It is seen that the new unknown quantities, which are regarded as the functions of c, increase very smoothly at the initial stage. In solving the Cauchy problem (26)–(30), the derivatives $\dot{a}_n(t)$ are determined by formula (24).

The replacement $t \to c$ is justified only under the condition that $dt/dc > 0$, which is obviously satisfied for small c. In numerical calculations, the sign of the right-hand side in (26) should be monitored and one should stop the calculation if $Q = 0$, which corresponds to an indefinitely high velocity of expansion of the wetted part of a body. The unlimited growth of the derivative dt/dc means that the velocity of the contact points is decreased and can change the direction. Thus, the contact region decreases in dimension, which indicates a partial exit of the body from water. Both cases, $dt/dc \to 0$ and $dt/dc \to \infty$, are of undoubtful interest in connection with a study of the influence of elastic effects on the process of impact on the liquid surface.

In a numerical solution of the Cauchy problem (26)–(30), the finite number of normal modes N is preserved, and it is assumed that $a_n \equiv 0$, $d_n \equiv 0$ for $n \geqslant N + 1$. Comparing the calculation results obtained for various N, one can draw a conclusion on the number of modes that is sufficient for estimation of the beam deflection, the velocity of its elements, and the distribution of bending stresses. For each of the enumerated characteristics, the number of modes permitting one to calculate it with required accuracy is particular. Strictly speaking, it is impossible to calculate the pressure in the contact region by the normal modes method.

To explain the last statement and to give instructions concerning the choice of the quantity N, we shall consider the hydrodynamic part of the initial problem separately. The solution of the problem for the Laplace equation (5) in the lower half-plane with the mixed boundary conditions (6) and (7) allows one to determine the horizontal velocity component in a liquid along the contact region [10]:

$$\varphi_x(x, 0, t) = \frac{1}{\pi\sqrt{c^2 - x^2}} \,\text{V.p.} \int\limits_{-c}^{c} \frac{\sqrt{c^2 - \sigma^2}}{\sigma - x} \varphi_y(\sigma, 0, t)\, d\sigma \qquad (-c < x < c). \tag{31}$$

The abbreviation V.p. means the Cauchy principal value of the integral. The derivative $\varphi_x(x, 0, t)$ has integrable singularities in the neighborhood of the contact points. With allowance for the expansion

$$\frac{\sqrt{c^2-\sigma^2}}{\sigma-x} = -\frac{\sigma}{\sqrt{c^2-\sigma^2}} - \frac{x}{\sqrt{c^2-\sigma^2}} + \frac{c^2-x^2}{(\sigma-x)\sqrt{c^2-\sigma^2}}$$

and the evenness of the vertical velocity component $\varphi_y(x,0,t)$ in x, formula (31) can be presented as follows:

$$\varphi_x(x,0,t) = -\frac{x}{\pi\sqrt{c^2-x^2}} \int_{-c}^{c} \frac{\varphi_y(\sigma,0,t)}{\sqrt{c^2-\sigma^2}} d\sigma + \frac{1}{\pi}\sqrt{c^2-x^2}\,\mathrm{V.p.} \int_{-c}^{c} \frac{\varphi_y(\sigma,0,t)}{(\sigma-x)\sqrt{c^2-\sigma^2}} d\sigma. \tag{32}$$

In the contact region, the derivative $\varphi_y(x,0,t)$ is limited, and the second integral in (32) therefore takes finite values for $-c \leqslant x \leqslant c$. In particular, as $x \to c-0$ it follows from (32) the asymptotic formula

$$\varphi_x(x,0,t) = -\frac{2x}{\pi\sqrt{c^2-x^2}} \int_0^{\pi/2} \varphi_y(c\sin\theta,0,t)\,d\theta + O(\sqrt{c^2-x^2}) \quad (-c < x < c) \tag{33}$$

with the separated singularity. The integral in (33) is calculated by means of conditions (7) and the representation (12):

$$\int_0^{\pi/2} \varphi_y(c\sin\theta,0,t)\,d\theta = -\frac{\pi}{2} + æ \sum_{n=1}^{\infty} \dot a_n(t)\Gamma_n(c).$$

Its value coincides to within constant factor with the denominator in (27). This means that the singularity of the horizontal velocity component of the liquid particles near the contact points vanishes as $Q \to \infty$, i.e., when the wetted part of the body begins to decrease.

The asymptotic formula (33) takes the form

$$\varphi_x(x,0,t) = A(t)\frac{x}{\sqrt{c^2-x^2}} + O(\sqrt{c^2-x^2}), \quad A(t) = 1 - \frac{2}{\pi}æ \sum_{n=1}^{\infty} \dot a_n(t)\Gamma_n(c).$$

Similarly, for the velocity potential and the pressure distribution in the contact region, we have

$$\begin{aligned} \varphi(x,0,t) &= -A(t)\sqrt{c^2-x^2} + O([c^2-x^2]^{3/2}), \\ p(x,0,t) &= \frac{c\dot c A(t)}{\sqrt{c^2-x^2}} + O([c^2-x^2]^{1/2}) \qquad (-c < x < c). \end{aligned} \tag{34}$$

The separated singularity of the velocity potential allows one to find the asymptotic behavior of the functions $b_n(t)$ as $n \to \infty$. Substituting (34) into (17), we obtain

$$b_n(t) = -A(t) \int_{-c}^{c} \sqrt{c^2-x^2}\psi_n(x)dx + \dots.$$

The discarded terms are the integrals of the smoother functions and, hence, contribute to the higher order of smallness as $n \to \infty$. For a simply supported beam, we have

$$b_n(t) = -A(t)f_n(c)[1+o(1)] \tag{35}$$

and $b_n(t) = O(n^{-3/2})$ as $n \to \infty$. It is possible to show that the last asymptotic formulas are valid with any attachment of the beam ends. Differentiating (35) with respect to t, we find $\dot b_n(t) = O(n^{-1/2})$ as $n \to \infty$, and hence the series for pressure in (16) converges only conditionally. Therefore, it is difficult to determine the pressure within the framework of the normal modes method.

At the same time, pressure is the important characteristic of the impact process. If the calculations show that the pressure is negative on the greater part of the contact region, it is impossible to consider further calculations to be adequate for an actual situation. In this case, cavitational phenomena should be taken into consideration. The following approach is proposed to calculate the pressure distribution along the contact region: the velocity potential of the wetted part of the plate is first calculated by (16), the asymptotic formula (35) being used for improving the convergence of a series; the pressure is then calculated by numerical differentiation with respect to t with allowance for the Cauchy–Lagrange integral (4) and asymptotic formulas (34).

Method of Numerical Solution of the Problem. With allowance for (12)–(15), the Euler equation (1) gives

$$\alpha\ddot{a}_n + \beta\lambda_n^4 a_n = \int_{-1}^{1} p(x,0,t)\psi_n(x)\,dx. \qquad (36)$$

If the right-hand sides in (36), where $n = 1, 2, 3, \ldots$, are known, then each equation describes forced oscillations of the system the period of eigenoscillations of which is equal to $t_n = 2\pi\lambda_n^{-2}(\alpha/\beta)^{1/2}$. In a numerical solution of (36), for $n = N$ the time step is Δt, and hence it should be much less than t_N. With increase in N, the quantity t_N decreases rapidly, and $t_N = O(N^{-2})$ as $N \to \infty$. Thus, the number of equations remaining in the system (26)–(29) in its numerical analysis cannot be arbitrarily great. It follows from (26) that the step in c should be of the order $\Delta c = \min(1/Q)\Delta t$, where the value itself of $\min(1/Q)$ depends on the solution and is not known beforehand. It would be evident to solve numerically system (26)–(29) with a variable step in c, which is determined on the basis of the value of the right-hand side in (26) at the previous step. We note that the case $Q \to 0$ does not lead to a decrease in the step in c.

However, more widespread are numerical schemes with a constant step relative to the independent variable. With a constant velocity of the impact, and here we consider only this case, it is difficult to expect that the wetted part of a body will decrease after its initial growth. To estimate the value of $\min(1/Q)$, we take $c(t) \approx c_r(t)$, where $c_r(t)$ is the dimension of the wetted part of the plate, ignoring elastic deformations. It is known [6] that $c_r(t) = 2t^{1/2}$. From here $Q = c/2$ and $\min(1/Q) = 2$ for $0 \leqslant c \leqslant 1$. We assume that $\Delta c = \Delta t$; even with a decrease in the rate of expansion of the contact region by a factor of two, the condition for the step Δc is satisfied. The more so if $\dot{c}(t) > \dot{c}_r(t)$.

The Cauchy problem (26)–(30), in which N normal modes are preserved, $a_n \equiv 0$, and $d_n \equiv 0$ for $n \geqslant N+1$, is solved numerically by the fourth order Runge–Kutta method with the step $\Delta c = 0.01 \cdot 2^{-M}$, where M is an integer such that $\Delta c \leqslant t_N/L_*$ and $M \geqslant 1$. The quantity L_* is equal to the number of the points which should be uniformly distributed on the interval $(0, 2\pi)$ for a sufficiently accurate representation of the function $\sin x$ by its values in these points; the approximate value of the function between the control points is determined by the method of quadratic interpolation. In the calculations, it was assumed that $L_* = 40$, 20, 10, and 5. If one assumes, for reasons of practice, that Δc cannot be less than 10^{-4}, the limiting number of modes N_+ is such that

$$\lambda_{N_+} \leqslant 100(2\pi/L_*)^{1/2}(\alpha/\beta)^{1/4}. \qquad (37)$$

For example, in the calculations of [4], where $\alpha = 4.88 \cdot 10^{-2}$ and $\beta = 1.8 \cdot 10^{-2}$, we have $N_+ = 32$ for $L_* = 10$. The number N_+ increases as the step Δc and/or the value L_* decreases, but the calculations can become less accurate. It follows from (37) that numerical calculations for small α (the wave impact on a plate made from a light material or of small thickness) are not effective. But it conflicts with the fact that system (26)–(29) for $\alpha = 0$ is not degenerate if $c > 0$. Here we limit ourselves to a physical explanation of this conflict. For $æ = 1$, the parameter α is equal to $M_B/(\rho L)$ and is proportional to the ratio of the total mass of a beam to the added mass of its high-frequency oscillations on the liquid surface. For light plates ($\alpha \ll 1$), the added mass is more important than their own mass and, hence, the form of Eqs. (36) does not correspond to the physics of the process. In this case, it is necessary to explicitly take into account the dependence of the right-hand sides in (36) on the generalized coordinates and their derivatives. Condition (37) does not hold true for plates made from light materials. For $\alpha \ll 1$ and small c, an asymptotic analysis of the initial problem (1)–(8) is necessary. Generally, condition (37) is sufficient but not necessary.

Condition (37) imposes the restriction on the maximum number of modes which can be preserved in system (26)–(29). However, the number of modes cannot be small, because it should be sufficient to represent the deflection and stresses in the beam to within good accuracy. The asymptotical behavior of the generalized

$a_n(t)$ coordinate as $n \to \infty$ and for $t > 0$ is determined by the smoothness of the pressure profile on the plate as a function of the variables x and t. According to (34), the term of the asymptotics of pressure relative to the smoothness has the form

$$p(x,0,t) = \frac{G(t)}{\sqrt{c^2 - x^2}} H(c^2 - x^2) + \dots, \tag{38}$$

where $H(x)$ is the Heaviside function, $G(0) \neq 0$, and $c(t) = O(t^{1/2})$ as $t \to 0$. It is assumed that $G(t)$ and $c(t)$ are smooth functions for $t > 0$. The term of the asymptotics in (38) has the same form as in the case of an undeformable plate, but the functions $G(t)$ and $c(t)$ in (38) are unknown beforehand. For a rigid plate, we have $G(t) = 2$ and $c(t) = 2\sqrt{t}$ [6]. We denote the right-hand side in (36) by $\alpha p_n(t)$. For the zero initial conditions, the solution of Eq. (36) is of the form

$$a_n(t) = \frac{1}{\omega_n} \mathrm{Im}\Big[\exp(i\omega_n t) \int\limits_0^t p_n(\tau) \exp(-i\omega_n \tau)\, d\tau\Big], \tag{39}$$

where $\omega_n^2 = \beta\lambda_n^4/\alpha$. We divide the integration interval in (39) into two parts $(0, \varepsilon_n)$ and (ε_n, t), where $\varepsilon_n \to 0$ and $\omega_n \varepsilon_n \to \infty$ as $n \to \infty$, and first consider the first integral. For small times ($0 < \tau < \varepsilon_n$), the elasticity of the plate can be ignored, and we have

$$\int\limits_0^{\varepsilon_n} p_n(\tau) \exp(-i\omega_n \tau)\, d\tau \sim 2\pi \int\limits_0^{\varepsilon_n} J_0(2\lambda_n\sqrt{\tau}) \exp(-i\omega_n \tau)\, d\tau$$

for large n. The replacement of the integration variable $\tau = \sigma/\omega_n$ allows one to write the last relation in the more convenient form

$$\int\limits_0^{\varepsilon_n} p_n(\tau) \exp(-i\omega_n \tau)\, d\tau \sim \frac{2\pi}{\omega_n} \int\limits_0^{\varepsilon_n \omega_n} J_0(\mu\sqrt{\sigma}) \exp(-i\sigma)\, d\sigma,$$

where $\mu = 2(\alpha/\beta)^{1/2}$. The last integral has a finite limit as $\varepsilon_n \omega_n \to \infty$, and hence

$$\int\limits_0^{\varepsilon_n} p_n(\tau) \exp(-i\omega_n \tau)\, d\tau \sim \frac{2\pi}{\omega_n} \exp[i(\mu^2/4 - \pi/2)] \qquad (n \to \infty).$$

Thus, the contribution of the initial stage of impact, when $t \ll 1$, to the asymptotics of the coefficients $a_n(t)$ has the order $O(n^{-4})$ as $n \to \infty$. Accordingly, $\dot{a}_n(t) = O(n^{-2})$ and $\ddot{a}_n(t) = O(1)$, as follows from (39).

Before analyzing the integral in (39) on the interval (ε_n, t), it is noteworthy that the asymptotics of the coefficients $p_n(t)$ as $n \to \infty$ is determined by the term in formula (38):

$$p_n(t) \sim 2\pi G(t) J_0[\lambda_n c(t)].$$

For $t > \varepsilon_n$, we have $\lambda_n c(t) > \lambda_n c(\varepsilon_n)$, where $c(\varepsilon_n) \sim 2\sqrt{\varepsilon_n}$ for $\varepsilon_n \ll 1$. It follows that $\lambda_n c(\varepsilon_n) = O(\sqrt{\varepsilon_n \omega_n})$ and tends to infinity as $n \to \infty$. Using the asymptotic expansion of the Bessel function $J_0(x)$ for large values of the argument, we find

$$p_n(t) \sim \frac{2\sqrt{2\pi}}{\sqrt{\lambda_n c}} G(t) \cos(\lambda_n c(t) - \pi/4),$$

$$\int\limits_{\varepsilon_n}^t p_n(\tau) \exp(-i\omega_n \tau)\, d\tau \sim \sqrt{\frac{2\pi}{\lambda_n}} \Big\{ \exp(-i\pi/4) \int\limits_{\varepsilon_n}^t \frac{G(\tau)}{\sqrt{c(\tau)}} \exp(i[\lambda_n c(\tau) - \omega_n \tau])\, d\tau$$

$$+ \exp(i\pi/4) \int\limits_{\varepsilon_n}^t \frac{G(\tau)}{\sqrt{c(\tau)}} \exp(-i[\lambda_n c(\tau) + \omega_n \tau])\, d\tau \Big\}.$$

We restrict ourselves to an analysis of the first integral

$$I_n(t) = \int\limits_{\varepsilon_n}^{t} \frac{G(\tau)}{\sqrt{c(\tau)}} \exp\left(-i[\omega_n \tau - \lambda_n c(\tau)]\right) d\tau$$

for $n \to \infty$; the second integral is considered similarly. To determine the order of smallness $I_n(t)$ as $n \to \infty$ and for $t > \varepsilon_n > 0$, we integrate by parts. We have

$$I_n(t) = i\left[\frac{G(\tau)\sqrt{c(\tau)}}{c(\tau)\omega_n - \dot{c}c\lambda_n} \exp\left(-i[\omega_n\tau - \lambda_n c(\tau)]\right)\right]_{\varepsilon_n}^{t} - i\int\limits_{\varepsilon_n}^{t} \frac{d}{d\tau}\left\{\frac{G(\tau)\sqrt{c(\tau)}}{c(\tau)\omega_n - \dot{c}c\lambda_n}\right\} \exp\left(-i[\omega_n\tau - \lambda_n c(\tau)]\right) d\tau.$$

Here the integral and out-of-integral terms for $\tau = \varepsilon_n$ have the same order $O(\omega_n^{-1}\varepsilon_n^{-1/4})$, and, for $\tau = t$, the order of the out-of-integral term is equal to $O(\omega_n^{-1})$. Hence, $I_n(t) = O(n^{-2+\gamma})$ as $n \to \infty$; as for the value of γ is concerned, it is only known that it is positive and does not exceed 0.5. As a result, we obtain

$$a_n(t) \sim -\frac{\pi\mu^2}{2\lambda_n^4} \cos\left(\omega_n t + \frac{1}{4}\mu^2\right) \qquad (n \to \infty). \tag{40}$$

For large numbers, the asymptotical behavior of the generalized $a_n(t)$ coordinates is determined by the conditions of the onset of the impact, when the pressure singularities in the contact points merge, and the pressure itself is unbounded, as follows from (38). We note that, as $n \to \infty$, the order of smallness of the following term in the asymptotic formula (40) exceeds little the order of the term, and therefore, from the viewpoint of practical evaluation, formula (40) cannot be used for better convergence of series (12). Because it is required to determine not only the deflection of the plate, but also the velocities of its points and the bending stresses, the series for which converge slowly by virtue of (40), the number of modes N preserved in system (26)–(30) should be sufficiently large, but not more than N_+, determined by inequality (37). The calculations were carried out for various N to establish the convergence of the numerical solution with increasing N.

The Cauchy problem (26)–(30) is solved by the Runge–Kutta method with a constant step in c. In each step, it is necessary to calculate the elements of the matrix S, determined by formula (23). The very possibility of the solution of the initial problem by the normal modes method is mainly determined by how effectively the functions $S_{nm}(c)$, where $0 < c \leqslant 1$, can be calculated. It turns out that these functions can be expressed by the zero- and first-order Bessel functions:

$$\begin{gathered} S_{nm}(c) = \frac{\pi c}{\lambda_n^2 - \lambda_m^2}[\lambda_n J_0(\lambda_m c)J_1(\lambda_n c) - \lambda_m J_0(\lambda_n c)J_1(\lambda_m c)] \qquad (n \neq m), \\ S_{nn}(c) = \frac{\pi}{2}c^2[J_0^2(\lambda_n c) + J_1^2(\lambda_n c)]. \end{gathered} \tag{41}$$

Polynomial approximations for the Bessel functions to within 10^{-7} are well known [11], and the necessity of repeated calculations of the matrix S does not, therefore, impose significant restrictions on the number of modes to be preserved. It is of interest to note that all the elements of the matrix S with finite numbers have the same asymptotical behavior for small c, namely, $S_{nm}(c) \sim \pi c^2/4$ for $c \to 0$.

Numerical Results. To show the distinguishing features of the proposed algorithm, system (26)–(29) was solved numerically for the following values of the parameters: $L = 0.5$ m, $R = 10$ m, $h = 2$ cm, $E = 21 \cdot 10^{10}$ N/m^2, $V = 3$ m/sec, $\rho = 1000$ kg/m^3, $\rho_b = 7850$ kg/m^3, and $b = 0.5$ m. Here ρ_b is the density of the material of the plate and b is its width. The total mass of the plate of size 1×0.5 m and thickness 2 cm is equal to 78.5 kg and coincides in magnitude with M_B. For $æ = 1$, we have $\alpha = 0.314$ and $\beta = 0.311$. The scale of beam deflection equals 2.5 cm, the scale of bending stresses in the plate is 420 N/mm^2, the scale of pressure is 0.18 N/mm^2, and the scale of time is 0.008 sec. The plate with the cited parameters but of smaller thickness was used in experiments [7] dealing with the influence of the elastic properties of a body on the process of water impact. The experimental conditions differed from those described above. In particular, in the experiment the plate thickness was 8 mm. For this value, the quantities α and β are comparable with the linearization parameter $L/R = 0.05$, and problem (1)–(8) can be investigated by asymptotic methods.

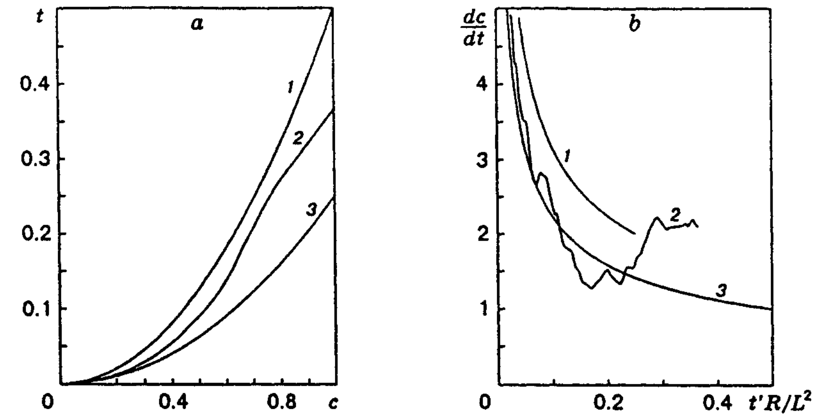

Fig. 2

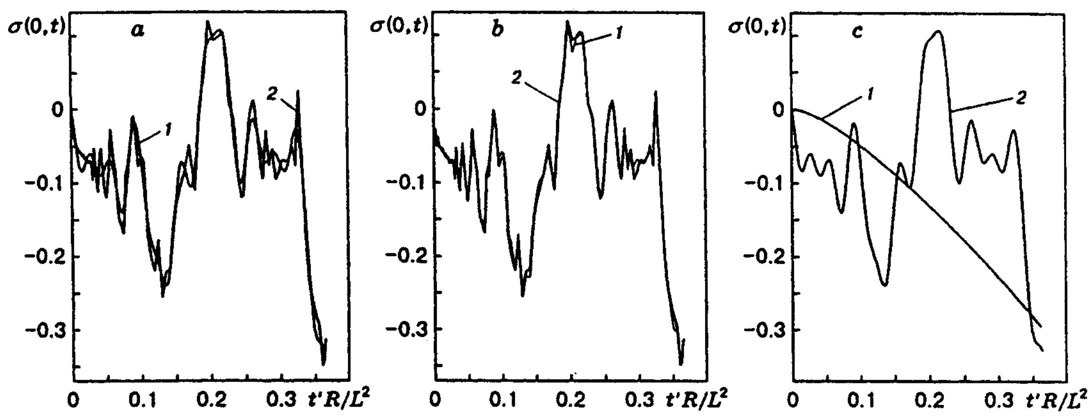

Fig. 3

In addition to the direct solution of problem (1)–(8), we consider two approximate approaches within the framework of which the function $c(t)$ was determined, ignoring the flexibility of the plate. The deformation of the free boundary of the liquid on impact was included in the first approach (the Wagner approach) and was not included in the second approach (the Kàrmàn approach). In the Kàrmàn approach, the contact points coincide with the points of intersection of the undeformable plate, $y = 0$, and $-1 < x < 1$; for a moving undeformable boundary of the liquid, we have $y = \varepsilon(-x^2/2 + t)$, and hence $c_K(t) = (2t)^{1/2}$, where $x = \pm c_K(t)$ sets the position of the points of intersection. Within the framework of the Wagner approach, the position of the contact points $[x = \pm c_W(t)]$ is determined by Eq. (11), where one can set $w(x,t) \equiv 0$ in the case of an undeformable plate. It follows that $c_W(t) = 2t^{1/2}$. Both approximate approaches require the solution of the Cauchy problem (26)–(30), where it is necessary to set $Q = c/2$ instead of (27) for the Wagner approach and $Q = c$ for the Kàrmàn approach. The quantities calculated within the framework of the Wagner approximate approach have the subscript W, and those calculated within the framework of the Kàrmàn approach have the subscript K.

The basic calculations were performed for $L_* = 20$ and $N = 15$. The impact stage t_* during which the plate is wetted only partially and $c(t_*) = 1$ is found equal to 0.36544. For comparison, we note that $t_{*W} = 0.25$ and $t_{*K} = 0.5$. It is seen that the duration of the impact stage for a rigid plate is approximately 1.5 times shorter than that for an elastic plate. The flexibility of the plate increases the duration of the impact stage. The maximum deviations of the function $t(c)$, $0 \leqslant c \leqslant 1$, calculated for $N = 1$, 5, and 10, from its values for $N = 15$ are less than $1.8 \cdot 10^{-2}$, $1.1 \cdot 10^{-4}$, and $2.1 \cdot 10^{-5}$, respectively. Figure 2a shows the functions $t_K(c)$, $t(c)$, and $t_W(c)$ (curves 1–3, respectively). Clearly, $c_K(t) < c(t) < c_W(t)$. For small t, the functions $c(t)$ and $c_W(t)$ are close to each other, which indicates the possibility to ignore the elasticity of the plate at the initial stage of impact. The velocity of the contact point $\dot{c}(t)$ is calculated for various N of the preserved modes. The maximum deviations of the function $\dot{c}(t)$, calculated for $N = 1, 5$, and 10, from its values calculated for $N = 15$ are less than 0.53, 0.12, and 0.03, respectively. The calculation of the velocity $\dot{c}(t)$ requires a larger number of modes compared to the calculation of the function $c(t)$. The functions $\dot{c}_W(t)$, $\dot{c}(t)$, and $\dot{c}_K(t)$ (curves 1–3, respectively) are shown in Fig. 2b. We note that the velocity of the contact point tends to infinity as $t \to 0$. The initial interval, on which the velocity of the contact points does not depend on the elastic properties of the plate, is very small. It is of interest to note that $\dot{c}(t)$ is close to $\dot{c}_K(t)$ at the initial stage. The evolution of the bending stresses in the center of the plate is shown in Fig. 3a for $N = 5$ and 15 (curves 1 and 2), in Fig. 3b for $N = 10$ and 15 (curves 1 and 2), and in Fig. 3c for $N = 1$ and 5 (curves 1 and 2). The maximum absolute values of the bending stresses are reached shortly before the end of the impact stage and equal approximately 0.33, which corresponds to approximately 140 N/mm^2 in dimensional variables. It is important to note that the single-mode approximation with $N = 1$, which is often used in practical calculations, gives no correct idea of the evolution of bending stresses during the impact stage, but allows one to estimate these stresses at the end of the stage.

Conclusion. In the present paper, a numerical algorithm of calculating the deformations of an elastic plate in its impact on the curved surface of a liquid has been given. The algorithm allows one to perform PC computations. The test calculations have shown the effectiveness of the algorithm. The algorithm is intended for analysis of the role of elastic effects in the impact processes of a liquid and thin-walled structures of limited extension.

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